

# Exact statistical properties of the zeros of complex random polynomials

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The zeros of complex Gaussian random polynomials, with coefficients such that the density in the underlying complex space is uniform, are known to have the same statistical properties as the zeros of the coherent state representation of one-dimensional chaotic quantum systems. We extend the interpretation of these polynomials by showing that they also arise as the wave function for a quantum particle in a magnetic field constructed from a random superposition of states in the lowest Landau level. A study of the statistical properties of the zeros is undertaken using exact formulas for the one and two point distribution functions. Attention is focussed on the moments of the two-point correlation in the bulk, the variance of a linear statistic, and the asymptotic form of the two-point correlation at the boundary. A comparison is made with the same quantities for the eigenvalues of complex Gaussian random matrices.

## 1 Introduction

The zeros of random polynomials have received recent attention as providing the universality class for the zeros of the coherent state representation of one-dimensional chaotic quantum systems [1]-[6]. Moreover, denoting the coefficients by  $\alpha_j$  so the polynomial reads

$$p(z) = \sum_{j=0}^N \alpha_j z^j, \quad (1.1)$$

it has recently been realized that in the particular case that the real and imaginary parts of  $\alpha_j$  are independent Gaussian random variables with mean zero and standard deviation  $\sigma_j$ , which we write as

$$\alpha_j = N[0, \sigma_j] + iN[0, \sigma_j], \quad (1.2)$$

the zero distribution is a solvable model [7] - [14]. Thus the probability density function (p.d.f.) can be computed in terms of the  $\sigma_j$  [7], as can the  $n$ -particle distribution functions for each  $n = 1, 2, \dots$  [9].

Our study is motivated by the explicit form of the zeros p.d.f.,

$$p(z_1, \dots, z_N) = \pi N! \left( \prod_{l=0}^N \frac{1}{\pi \sigma_l^2} \right) \frac{\prod_{1 \leq j < k \leq N} |z_k - z_j|^2}{(\sum_{j=0}^N |e_j|^2 / \sigma_{N-j}^2)^N}, \quad (1.3)$$

where

$$e_j := \sum_{1 \leq p_1 < \dots < p_j \leq N} z_{p_1} \cdots z_{p_j} \quad (1.4)$$

and  $e_0 := 1$ . As pointed out in [9], the factor

$$\prod_{1 \leq j < k \leq N} |z_k - z_j|^2 = e^{-2 \sum_{j < k} \log |z_k - z_j|} \quad (1.5)$$

can be interpreted as the Boltzmann factor for a two-dimensional classical  $N$  particle system with pairwise logarithmic repulsion at inverse temperature  $\beta = 2$ . The particles are prevented from repelling to infinity by an extensive many body interaction with Boltzmann factor  $(\sum_{j=0}^N |e_j|^2 / \sigma_{N-j}^2)^{-N}$ . Furthermore, with the particular standard deviation

$$\sigma_j = \frac{1}{(j!)^{1/2}} \quad (1.6)$$

the density of zeros is, to leading order, uniform inside a disk of radius  $\sqrt{N}$  and zero outside this disk [5] (see also Section 5). This latter feature is shared by the eigenvalue p.d.f. for complex random matrices with each element independent and chosen from the complex Gaussian distribution  $N[0, 1/\sqrt{2}] + iN[0, 1/\sqrt{2}]$ , which is proportional to

$$\prod_{j=1}^N e^{-\sum_{j=1}^N |z_j|^2} \prod_{1 \leq j < k \leq N} |z_k - z_j|^2. \quad (1.7)$$

Let us consider further the eigenvalue p.d.f. (1.7). Some insight into the behaviour of the density can be obtained by noting that for  $|z| < \sqrt{N}$ , (1.7) is proportional to the Boltzmann factor of the two-dimensional one-component plasma (2dOCP) at the coupling  $\beta = 2$ . The factor  $\exp(-\sum_{j=1}^N |z_j|^2)$  there results from the potential energy between a uniform background of charge density  $-1/\pi$  and  $N$  unit charges interacting via the logarithmic potential, so the fact that the density has support in a disk of radius  $\sqrt{N}$  can be interpreted as the system having a preference for charge neutrality. Moreover, the plasma system is an example of a two-dimensional Coulomb system in its conductive phase, and so exhibits universal properties for the behaviour of the two-point correlation, both in the bulk and at the boundary  $|z| \approx \sqrt{N}$  (see e.g. [15, 16]). These special forms in turn imply a special form for the variance of a linear statistic [17]. The eigenvalue distribution of course must also exhibit these Coulomb properties. Because the zeros p.d.f. has the factor (1.5) in common with (1.7), and with the choice (1.6) has the density in common too, we are led to investigate for the zeros distribution the two-point correlation in the bulk and at the boundary  $|z| \approx \sqrt{N}$ , and the variance of a linear statistic, to see if they exhibit universal Coulomb like properties.

The plan of the paper is as follows. In Section 2 an interpretation of the complex Gaussian random polynomial with variance (1.6) will be given as a random wave function associated with

a superposition of lowest Landau states in the plane. A similar interpretation will be given to the complex Gaussian random polynomial with variance of the coefficients

$$\sigma_j^2 = \binom{N}{j}, \quad (1.8)$$

considered originally in [9], except this time the Landau states are those for a particle confined to the surface of the sphere with a perpendicular magnetic field. In Section 3 the known [9] general formulas for the one and two-point distributions are revised, and these are specialized to the cases (1.6) and (1.8). The investigation proper is carried out in Sections 4 and 5. In Section 4 we study bulk properties of the two-point correlation function, and show how the formulas obtained for its moments have consequence with regard to fluctuation formulas for the number of particles in a region  $\Lambda$  as  $|\Lambda| \rightarrow 0$ , and the variance of a linear statistic as a function of  $N$  in the sphere case (1.8). The behaviour of the density and two-point correlation at the boundary of support in the disk case (1.6) is the subject of Section 5. The paper ends with a brief summary of our results.

## 2 Quantum mechanical interpretation

A number of studies [1]-[6] have related complex polynomials to the wave function of chaotic quantum systems. More precisely, complex random polynomials with variances (1.6) and (1.8) arise as the Bargmann or coherent state representation of one-dimensional chaotic wave functions and chaotic spin states respectively. In this section we point out that these complex random polynomials also occur as random superpositions of lowest Landau level states for a charged quantum particle, in a plane and on the surface of a sphere, in the presence of a perpendicular magnetic field.

Consider first the planar problem. Let  $B$  denote the strength of the magnetic field, and suppose the vector potential  $\vec{A}$  is chosen in the symmetric gauge so that

$$\vec{A} = \frac{B}{2}(-y\hat{x} + x\hat{y}).$$

Then it is well known (see e.g. [18]) that the lowest Landau level (i.e. ground state) consists of any wave function of the form

$$e^{-|z|^2/4\ell^2} f(\bar{z}), \quad (2.1)$$

where  $z := x + iy$ ,  $\ell := \sqrt{\hbar c/eB}$  ( $e$  denotes the charge of the particles) is the magnetic length and  $f$  is analytic in  $\bar{z}$ . Furthermore the states (2.1) can be separated by requiring that they be simultaneous eigenstates of the quantum mechanical operator corresponding to the square of the centre of the cyclotron orbit. This operator has eigenvalues  $R_n^2 = (2n+1)\ell^2$  ( $n = 0, 1, \dots$ ) with corresponding normalized eigenfunctions of the form (2.1) given by

$$\psi_n(\vec{r}) = \frac{\bar{z}^n e^{-|z|^2/2}}{\pi^{1/2} (n!)^{1/2}} \quad (2.2)$$

(here the units have been chosen so that  $2l^2 = 1$ ). A state  $\phi(\vec{r})$  consisting of the first  $N + 1$  of these states has the form

$$\phi(\vec{r}) = \frac{1}{\mathcal{N}\pi^{1/2}} e^{-|z|^2/2} p(\bar{z}), \quad p(w) := \sum_{n=0}^N \frac{\alpha_n}{\sqrt{n!}} w^n \quad (2.3)$$

with  $\mathcal{N} := (\sum_{j=0}^N |\alpha_n|^2)^{1/2}$ . Choosing the state to be random by selecting each coefficient  $\alpha_n$  at random from the complex Gaussian distribution  $\mathcal{N}[0, 1] + i\mathcal{N}[0, 1]$ , we see immediately that  $p(w)$  is a complex Gaussian random polynomial with variance (1.6).

For the sphere problem, the magnetic field is due to a monopole, which must satisfy [20] the quantization condition  $B = N\hbar c/2eR^2$ ,  $N = 1, 2, \dots$ . Furthermore the Hamiltonian operator  $H$  can be written [19]

$$H = \frac{1}{2mR^2} \left( \vec{L}^2 - \frac{\hbar^2 N^2}{4} \right), \quad (2.4)$$

where the components of  $L$  obey the canonical commutation relations for angular momentum

$$[L_x, L_y] = i\hbar L_z, \quad [L_y, L_z] = i\hbar L_x, \quad [L_z, L_x] = i\hbar L_y.$$

Hence the allowed values of  $\vec{L}^2$  are  $l(l+1)\hbar^2$ ,  $l = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots$  and so for (2.4) to be positive definite the smallest allowed value of  $l$  is  $N/2$ , which thus corresponds to the lowest Landau level. The states in this level can be distinguished by seeking simultaneous eigenfunctions of  $L_z$  and  $H$ . This shows that there is a  $N + 1$  fold degeneracy, which is spanned by the orthogonal functions

$$\psi_m(\theta, \phi) = \left\{ \frac{N/2 + 1}{4\pi R^2} \binom{N}{N/2 + m} \right\}^{1/2} u^{N/2+m} v^{N/2-m} e^{-iN\phi/2}, \quad m = -N/2, -N/2+1, \dots, N/2,$$

where

$$u := \cos \frac{\theta}{2} e^{i\phi/2}, \quad v = -i \sin \frac{\theta}{2} e^{-i\phi/2}$$

denote the Cayley-Klein parameters and  $(\theta, \phi)$  are the usual spherical coordinates. Forming a linear combination of such states gives the state

$$\Phi(\theta, \phi) = \frac{1}{\mathcal{N}} \left( \frac{N/2 + 1}{4\pi R^2} \right)^{1/2} (-i \sin \theta/2)^N e^{-iN\phi/2} p(e^{i\phi} \cot \frac{\theta}{2}), \quad p(z) = \sum_{n=0}^N \binom{N}{n}^{1/2} \alpha_n z^n.$$

If this state is chosen at random by specifying the  $\alpha_n$  as belonging to the random complex Gaussian distribution  $\mathcal{N}[0, 1] + i\mathcal{N}[0, 1]$  as in (2.3), then the polynomial  $p(z)$  is a complex random polynomial with variance (1.8). Furthermore, the mapping

$$z = e^{i\phi} \cot \frac{\theta}{2} \quad (2.5)$$

represents a stereographic projection from the north pole of the unit sphere to a plane passing through its equator, showing that  $p$  as a function of the spherical coordinates  $(\theta, \phi)$  has  $N$  zeros on the surface of the unit sphere. Such a factorization of a spin state is due to Majorana [21].

### 3 The distribution functions

As mentioned in the Introduction, the general  $n$ -point distribution function  $\rho_{(n)}$  for the zeros of the random polynomial (1.1) can be computed exactly in the case that the coefficients are complex Gaussian random variables drawn from the distribution (1.2) [9]. Our interest is in this explicit form in the cases  $n = 1$  and  $n = 2$ .

The case  $n = 1$  corresponds to the density of zeros. This one point distribution function has the explicit form [9, 14]

$$\pi\rho_{(1)}(\vec{r}) = \frac{\partial^2}{\partial z \partial \bar{z}} \log \langle p(z)p(\bar{z}) \rangle, \quad (3.6)$$

where the averages are with respect to the distribution of the coefficients. Here  $\vec{r} := (x, y)$  with  $x + iy := z$ . Performing this average in the case that the distribution of the coefficients  $\alpha_j$  is given by (1.6) gives

$$\pi\rho_{(1)}(\vec{r}) = \frac{\partial^2}{\partial z \partial \bar{z}} \log e(|z|^2; N) \quad (3.7)$$

where

$$e(u; n) := \sum_{p=0}^n \frac{u^p}{p!}. \quad (3.8)$$

Notice that with  $z$  fixed  $e(|z|^2; N) \rightarrow e^{|z|^2}$  as  $N \rightarrow \infty$ , so the formula (3.7) gives that

$$\rho_{(1)}(\vec{r}) = \frac{1}{\pi} \quad (3.9)$$

in the bulk.

Repeating the calculation which led to (3.7) with the coefficients given by (1.8) shows that the corresponding density is

$$\pi\rho_{(1)}(\vec{r}) = \frac{\partial^2}{\partial z \partial \bar{z}} \log \left( 1 + |z|^2 \right)^N. \quad (3.10)$$

Due to the physical origin of the variances (1.6), it is natural to perform a stereographic projection onto the sphere according to the mapping (2.5) so that

$$\rho_{(1)}(\vec{r}) dx dy = \rho_{(1)}(\hat{s}) \frac{4 dS}{(1 + |z|^2)^2}, \quad (3.11)$$

where  $\rho(\hat{s})$  refers to the density on the unit sphere with surface element  $dS$  at the point  $\hat{s}$  determined by (2.5) and  $4/(1 + |z|^2)^2$  is the Jacobian for the change of variables. This gives [8, 9, 14]

$$\rho_{(1)}(\hat{s}) = \frac{N}{4\pi}, \quad (3.12)$$

thus showing that the density is uniform.

Not surprisingly, the complexity of the explicit form of the  $n$ -point distribution increases with  $n$ . We will not go beyond the  $n = 2$  case, for which the distribution has the explicit form

$$\pi^2 \rho_{(2)}(\vec{r}_1, \vec{r}_2) = \frac{\text{per}(C - B^\dagger A^{-1} B)}{\det A}. \quad (3.13)$$

The notation  $\text{per}$  denotes the permanent, which for a  $2 \times 2$  matrix is defined by

$$\text{per} \begin{bmatrix} a & b \\ c & d \end{bmatrix} := ad + bc, \quad (3.14)$$

and with  $p_j := p(z_j)$ ,  $p'_j := p'(z_j)$ , ( $j = 1, 2$  and the dash denotes differentiation, the matrices  $A$ ,  $B$  and  $C$  are defined by

$$A := \begin{bmatrix} \langle p_1 \bar{p}_1 \rangle & \langle p_1 \bar{p}_2 \rangle \\ \langle p_2 \bar{p}_1 \rangle & \langle p_2 \bar{p}_2 \rangle \end{bmatrix}, \quad B := \begin{bmatrix} \langle p_1 \bar{p}'_1 \rangle & \langle p_1 \bar{p}'_2 \rangle \\ \langle p_2 \bar{p}'_1 \rangle & \langle p_2 \bar{p}'_2 \rangle \end{bmatrix}, \quad C := \begin{bmatrix} \langle p'_1 \bar{p}'_1 \rangle & \langle p'_1 \bar{p}'_2 \rangle \\ \langle p'_2 \bar{p}'_1 \rangle & \langle p'_2 \bar{p}'_2 \rangle \end{bmatrix}. \quad (3.15)$$

Writing

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} =: \frac{1}{\det A} A^D$$

we see that (3.13) can be written in the equivalent form

$$\pi^2 \rho_{(2)}(\vec{r}_1, \vec{r}_2) = \frac{\text{per}(dC - B^\dagger A^D B)}{d^3}, \quad d = \det A. \quad (3.16)$$

Consider now the evaluation of the matrix elements in (3.15) in the case of (1.6). These are made explicit by referring to the formulas

$$\begin{aligned} \langle p_i \bar{p}_j \rangle &= e(z_i \bar{z}_j; N) & \langle p'_i \bar{p}_j \rangle &= \bar{z}_j e(z_i \bar{z}_j; N-1) \\ \langle p_i \bar{p}'_j \rangle &= z_i e(z_i \bar{z}_j; N-1) & \langle p'_i \bar{p}'_j \rangle &= z_i \bar{z}_j e(z_i \bar{z}_j; N-2) + e(z_i \bar{z}_j; N-1), \end{aligned} \quad (3.17)$$

where  $e(u; n)$  is defined by (3.8), which in turn follow from the definitions. At this stage we will not attempt to further develop (3.16); this task will be undertaken in Section 5 when the  $N \rightarrow \infty$  limit in the neighbourhood of the boundary  $|z| = \sqrt{N}$  is computed. However we will note the previously computed [9]  $N \rightarrow \infty$  limiting value with  $\vec{r}_1$  and  $\vec{r}_2$  fixed. This limit gives the 2-point distribution function in the bulk, and has the explicit evaluation<sup>1</sup>

$$\rho_{(2)}(\vec{r}_1, \vec{r}_2) = \frac{1}{\pi^2} h(|\vec{r}_1 - \vec{r}_2|^2/2), \quad h(x) := \frac{(\sinh^2 x + x^2) \cosh x - 2x \sinh x}{\sinh^3 x}. \quad (3.18)$$

In the case of the variances (1.8) we also require knowledge of the matrix elements in (3.15). A simple calculation gives [9]

$$\begin{aligned} \langle p_i \bar{p}_j \rangle &= (1 + z_i \bar{z}_j)^N & \langle p'_i \bar{p}_j \rangle &= N \bar{z}_j (1 + z_i \bar{z}_j)^{N-1} \\ \langle p_i \bar{p}'_j \rangle &= N z_i (1 + z_i \bar{z}_j)^{N-1} & \langle p'_i \bar{p}'_j \rangle &= N (1 + N z_i \bar{z}_j) (1 + z_i \bar{z}_j)^{N-2}. \end{aligned} \quad (3.19)$$

In Section 4 we will have application for the corresponding evaluation of (3.16) in the finite system, when projected onto the sphere according to the transformation (2.5). By substituting (3.19) into (3.15), and then the result into (3.16), expanding the permanent according to (3.14) and performing some elementary but tedious manipulations we find

$$\begin{aligned} \rho_{(2)}^T(\hat{s}_1, \hat{s}_2) &:= \rho_{(2)}(\hat{s}_1, \hat{s}_2) - \rho_{(1)}(\hat{s}_1) \rho_{(1)}(\hat{s}_2) \\ &= \frac{N^2}{16\pi^2} \frac{f^{2N-4}}{1 - f^{2N}} \left( 1 + f^4 - 2N \frac{1 - f^4}{1 - f^{2N}} + N^2 \frac{(1 + f^{2N})(1 - f^2)^2}{(1 - f^{2N})^2} \right) \end{aligned} \quad (3.20)$$

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<sup>1</sup> In [9] the factor of 1/2 in the argument of  $h(|\vec{r}_1 - \vec{r}_2|^2/2)$  is missing; this has been subsequently corrected in [6].

where

$$f^2 = (f(\hat{s}_1, \hat{s}_2))^2 := \frac{1}{2}(1 + \hat{s}_1 \cdot \hat{s}_2). \quad (3.21)$$

## 4 The distribution functions in the bulk and fluctuation formulas

### 4.1 The two-point function in the bulk

In this section properties of the bulk two-point distribution function for the 2dOCP at  $\beta = 2$ , or equivalently for the eigenvalues of complex random matrices specified in the Introduction, will be recalled and contrasted with the corresponding properties of the bulk two-point distribution function (3.18). Now, for a bulk density  $\rho_{(1)}(\vec{r}) = 1/\pi$ , the bulk two-point distribution function for the 2dOCP at  $\beta = 2$  has the explicit form [22, 23]

$$\rho_{\text{OCP}(2)}(\vec{r}_1, \vec{r}_2) = \frac{1}{\pi^2} \left( 1 - e^{-|\vec{r}_1 - \vec{r}_2|^2} \right), \quad (4.1)$$

or equivalently the truncated distribution has the explicit form

$$\rho_{\text{OCP}(2)}^T(\vec{r}_1, \vec{r}_2) = -\frac{1}{\pi^2} e^{-|\vec{r}_1 - \vec{r}_2|^2}. \quad (4.2)$$

One feature of (4.1) is that it is a strictly negative monotonic function and is its own large separation asymptotic form,

$$\rho_{\text{OCP}(2)}^T(\vec{r}, \vec{0}) \underset{|\vec{r}| \rightarrow \infty}{\sim} -\frac{1}{\pi^2} e^{-|\vec{r}|^2}.$$

In contrast (3.18) reaches a maximum before asymptotically approaching zero from above (see e.g. the plot in [9]). The large separation asymptotic form is computed from (3.18) to be

$$\rho_{(2)}^T(\vec{r}, \vec{0}) \underset{|\vec{r}| \rightarrow \infty}{\sim} |\vec{r}|^4 e^{-|\vec{r}|^2}.$$

The zeroth and second moments of  $\rho_{\text{OCP}(2)}^T$  exhibit particular universal properties of Coulomb systems. The zeroth moment is evaluated as

$$\int_{\mathbb{R}^2} \rho_{\text{OCP}(2)}^T(\vec{r}, \vec{0}) d\vec{r} = -\frac{1}{\pi},$$

which can be written in the equivalent form

$$\int_{\mathbb{R}^2} \left( \rho_{\text{OCP}(2)}^T(\vec{r}, \vec{0}) + \rho_{(1)}(\vec{r}) \delta(\vec{r}) \right) d\vec{r} = 0. \quad (4.3)$$

In this guise the zeroth moment formula can be interpreted as an example of a charge sum rule for Coulomb systems (see e.g. [15]) which says the total charge of a fixed charge (here a charge at the origin) and its screening cloud (represented by  $\rho_{\text{OCP}(2)}^T(\vec{r}, \vec{0})$ ) is zero.

The second moment of (4.1) is evaluated as

$$\int_{\mathbb{R}^2} |\vec{r}|^2 \rho_{\text{OCP}(2)}^T(\vec{r}, \vec{0}) d\vec{r} = -\frac{1}{\pi}. \quad (4.4)$$

By invoking a linear response relation (see e.g. [15, 16]) this can be shown to be equivalent to the statement that the system perfectly screens an external charge density in the long wavelength limit. Thus suppose the 2dOCP for general  $\beta$  is perturbed by an external charge density  $\epsilon e^{i\vec{k}\cdot\vec{r}}$ . The potential energy of the system is then changed by an amount

$$\begin{aligned}\delta U &:= -\epsilon \int_{\mathbf{R}^2} d\vec{r}' \int_{\mathbf{R}^2} d\vec{r} \log |\vec{r} - \vec{r}'| n_{(1)}(\vec{r}') e^{i\vec{k}\cdot\vec{r}} \\ &= \frac{2\epsilon\pi}{|\vec{k}|^2} \tilde{n}_{(1)}(\vec{k}),\end{aligned}\tag{4.5}$$

where  $n_{(1)}(\vec{r})$  represents the microscopic density at the point  $\vec{r}$ ,  $\tilde{n}_{(1)}(\vec{k})$  its two-dimensional Fourier transform, and to obtain the second equality the result

$$-\int_{\mathbf{R}^2} d\vec{r} \log |\vec{r}| = \frac{2\pi}{|\vec{k}|^2}\tag{4.6}$$

from the theory of generalized functions has been used. Suppose now that the microscopic charge density at the point  $\vec{r}$ , which for the OCP is just  $n_{(1)}(\vec{r})$ , is observed. The linear response relation gives that

$$\langle n_{(1)}(\vec{r}) \rangle_\epsilon - \langle n_{(1)}(\vec{r}) \rangle_0 = -\beta \langle n_{(1)}(\vec{r}) \delta U \rangle_0^T,\tag{4.7}$$

where the subscript  $(\epsilon)$  indicates the system in the presence of the perturbation. Now, a characteristic of a Coulomb system in its conductive phase is that it will perfectly screen an external charge density in the long wavelength limit. This means that

$$\langle n_{(1)}(\vec{r}) \rangle_\epsilon - \langle n_{(1)}(\vec{r}) \rangle_0 \underset{|\vec{k}| \rightarrow 0}{\sim} -\epsilon e^{i\vec{k}\cdot\vec{r}}.$$

Substituting this in (4.7), and using the translational invariance of the system in the bulk, we see that

$$\tilde{S}(\vec{k}) \underset{|\vec{k}| \rightarrow 0}{\sim} \frac{|\vec{k}|^2}{2\pi\beta}, \quad S(\vec{r}) := \rho_{(2)}^T(\vec{r}, \vec{0}) + \rho\delta(\vec{r})\tag{4.8}$$

This implies the charge sum formula (4.3) as well as the second moment (Stillinger-Lovett) sum rule

$$\int_{\mathbf{R}^2} d\vec{r} |\vec{r}|^2 \rho_{\text{OCP}(2)}^T(\vec{r}, \vec{0}) = -\frac{2}{\beta\pi},\tag{4.9}$$

which reduces to (4.4) at  $\beta = 2$ .

Let us now compute the zeroth and second moments of (3.18), and compare their values with those for (4.1). Now, whereas the computation of the moments in the case of (4.1) is elementary, it is not so simple to do likewise for (3.18). Fortunately the function  $h(x)$  in (3.18) can be written as the second derivative of an elementary function. Thus by explicit calculation we can check that

$$h(x) = \frac{1}{2} \frac{d^2}{dx^2} x^2 \coth x,\tag{4.10}$$

or equivalently

$$h(x) - 1 = \frac{1}{2} \frac{d^2}{dx^2} \left( x^2 (\coth x - 1) \right).\tag{4.11}$$

Making use of (4.11) we find that again the charge sum rule (4.3) is satisfied, whereas for the second moment we find

$$\int_{\mathbf{R}^2} d\vec{r} |\vec{r}|^2 \rho_{(2)}^T(\vec{r}, \vec{0}) = 0. \quad (4.12)$$

Although we have know that when viewed as a classical particle system the p.d.f. (1.3) for the zeros distribution contains many body forces in addition to the pair potential term (1.5), it is of some interest to compare the results (4.3) and (4.12) to those for a two-dimensional classical fluid with purely pairwise interactions. For long-range potentials with Fourier transform  $\tilde{v}(\vec{k}) \sim c|\vec{k}|^{-\alpha}$ ,  $\alpha > 0$ , we can generalize the perfect screening argument which led to (4.8), this time obtaining

$$\tilde{S}(\vec{k}) \underset{|\vec{k}| \rightarrow 0}{\sim} \frac{|\vec{k}|^\alpha}{c\beta}. \quad (4.13)$$

Setting  $\vec{k} = \vec{0}$  gives (4.3) in all cases. However, as is well known (see e.g. [26]), it is only for even values of  $\alpha$  that (4.13) is compatible with a fast decay of  $S(\vec{r})$ . In the Coulomb case  $\alpha = 2$ , (4.13) then gives the second moment condition (4.9). The next smallest even value is  $\alpha = 4$ . Then (4.13) indeed implies that the second moment vanishes, just as observed in (4.12) for the complex zeros. However a pair potential in two-dimensions for which the Fourier transform behaves as  $|\vec{k}|^4$  for  $|\vec{k}| \rightarrow 0$ , has the large  $|\vec{r}|$  behaviour in real space proportional to  $-|\vec{r}|^2 \log |\vec{r}|$ . It is unlikely that a classical system with such strong repulsion can be thermodynamically stable, even with a neutralizing background.

## 4.2 Fluctuation formulas in the infinite system

In general the properties of the two-point function of a many particle system, in particular the value of its moments, have consequences regarding fluctuation formulas for linear statistics. We recall that a linear statistic is any observable (function) of the form  $A = \sum_{j=1}^N a(\vec{r}_j)$ . It is easy to see from the definitions that the two-point function  $S(\vec{r})$  (4.8) is directly related to the fluctuation (variance) of a linear statistic via the formula

$$\begin{aligned} \text{Var}(A) &:= \langle A^2 \rangle - \langle A \rangle^2 \\ &= \int d\vec{r}_1 a(\vec{r}_1) \int d\vec{r}_2 a(\vec{r}_2) S(\vec{r}). \end{aligned} \quad (4.14)$$

The linear statistic specified by  $a(\vec{r}) = \chi_\Lambda(\vec{r})$  where  $\chi_\Lambda$  denotes the indicator function of the domain  $\Lambda$ :  $\chi_\Lambda(\vec{r}) = 1$  if  $\vec{r} \in \Lambda$ ,  $\chi_\Lambda(\vec{r}) = 0$  otherwise, measures the number of particles in the region  $\Lambda$ . In this case  $\text{Var}(A)$  is the number variance, to be denoted  $\Sigma_2(\Lambda)$  which specified the fluctuation from the mean of the number of particles in  $\Lambda$ . It is a well known fact [26] that whenever  $\Lambda$  can be generated by a dilation from a fixed domain  $\Lambda_0$ , and the charge sum rule (4.3) holds, the number variance has the asymptotic behaviour

$$\lim_{|\Lambda| \rightarrow \infty} \frac{\Sigma_2(\Lambda)}{|\partial\Lambda|} = -\frac{1}{\pi} \int_{\mathbf{R}^2} d\vec{r} |\vec{r}| \rho_{(2)}^T(\vec{r}, \vec{0}), \quad (4.15)$$

assuming the integral exists. Here, with  $\Lambda$  a two-dimensional domain,  $|\partial\Lambda|$  denotes the length of the perimeter of  $\Lambda$ . Thus the fluctuations are strongly suppressed in comparison to a compressible fluid for which  $\Sigma_2(\Lambda)$  is proportional to  $|\Lambda|$ .

In the case of the complex zeros the two-point function is specified by (3.18). Making use of the formula (4.11) the first moment in (4.15) can be computed exactly. For this purpose, after introducing polar coordinates, we integrate by parts twice and make use of the definite integral

$$\int_0^\infty \frac{t^{1/2} e^{-t}}{1 - e^{-t}} dt = \frac{\sqrt{\pi}}{2} \zeta\left(\frac{3}{2}\right).$$

Substituting the result in (4.15) gives

$$\lim_{|\Lambda| \rightarrow \infty} \frac{\Sigma_2(\Lambda)}{|\partial\Lambda|} = \frac{1}{8\pi^{3/2}} \zeta\left(\frac{3}{2}\right) = 0.0586436 \dots, \quad (4.16)$$

where  $\zeta(s)$  denotes the Riemann zeta function. In the special case that  $\Lambda$  is a disk of radius  $r$ , Prosen [10] has obtained a one dimensional integral representation of  $\Sigma_2(\Lambda)$ , and from this obtained a numerical evaluation consistent with (4.16).

Another class of linear statistics of particular interest with regard to the eigenvalues of random matrices (see e.g. [17] and references therein) is when  $a(\vec{r})$  varies on macroscopic length scales relative to the spacing between particles. This can be achieved by first supposing  $a(\vec{r})$  varies on the length scale of the spacing between zeros, and then making the replacement  $a(\vec{r}) \mapsto a(\vec{r}/\alpha)$  where  $\alpha \gg 1$ . Now, writing  $S(\vec{r})$  in (4.14) in terms of its Fourier transform we have the formula

$$\text{Var}(A) = \frac{1}{(2\pi)^2} \int_{\mathbf{R}^2} d\vec{r} |\tilde{a}(k)|^2 \tilde{S}(\vec{k}).$$

Making the replacement  $a(\vec{r}) \mapsto a(\vec{r}/\alpha)$  then gives

$$\text{Var}(A) = \frac{\alpha^2}{(2\pi)^2} \int_{\mathbf{R}^2} d\vec{r} |\tilde{a}(k)|^2 \tilde{S}(\vec{k}/\alpha). \quad (4.17)$$

Since  $\alpha \gg 1$  we see that the small  $|\vec{k}|$  behaviour of  $\tilde{S}(\vec{k})$  determines  $\text{Var}(A)$ . For the OCP  $\tilde{S}(\vec{k})$  exhibits the quadratic behaviour (4.8) and so  $\text{Var}(A)$  is formally independent of  $\alpha$  in the macroscopic limit [16]. For the complex zeros, as already noted the fact that the zeroth and second moments of  $S(\vec{r})$  vanishes implies that

$$\tilde{S}(\vec{k}) \underset{|\vec{k}| \rightarrow 0}{\sim} c \vec{k}^4. \quad (4.18)$$

Furthermore, from the definitions it is easy to see that  $c$  is related to the fourth moment of  $\rho_{(2)}^T$  by

$$c = \frac{1}{64} \int_{\mathbf{R}^2} d\vec{r} \vec{r}^4 \rho_{(2)}^T(\vec{r}, \vec{0}).$$

Using (4.11) this integral can be computed exactly, giving

$$c = \frac{1}{8\pi} \zeta(3)$$

(in fact all coefficients in the power series expansion of  $\tilde{S}(\vec{k})$  have been calculated in [4]).

Substituting (4.18) in (4.17) we see that

$$\text{Var}(A) \underset{\alpha \rightarrow \infty}{\sim} \left( \frac{c}{(2\pi)^2} \int_{\mathbf{R}^2} d\vec{k} |\tilde{a}(k)|^2 |\vec{k}|^4 \right) \frac{1}{\alpha^2}, \quad (4.19)$$

assuming the integral exists. (Note that the integral does not exist for  $a(\vec{r}) = \chi_{\Lambda_0}$ , in keeping with the distinct behaviours of (4.15) and (4.19).)

### 4.3 Fluctuation formulas on the sphere

The fluctuation formula (4.19) applies to the infinite system. Also of interest is the situation of a fixed volume, and thus the density being proportional to  $N$  as the number of particles increases. This is the natural setting for the zeros of the complex random polynomial with variances specified by (1.6) projected onto the unit sphere. We know from (3.12) that the zeros are uniformly distributed. The question of interest is the behaviour of  $\text{Var}(A)$  as a function of  $N$ .

In fact this question, and its generalization to variance of linear statistics for zeros of multidimensional polynomials projected on to the surface of higher dimensional spheres for which the zero density is uniform, has been addressed recently by Shiffman and Zelditch [14]. They proved that for a statistic with  $a(\theta, \phi)$  smooth on the unit sphere,

$$\text{Var}(A) = O(1) \quad (4.20)$$

as  $N \rightarrow \infty$ . This bound was also established in the higher dimensional cases. The question now is if the  $O(1)$  behaviour is asymptotically exact. In light of the analysis for the infinite system presented in the previous subsection, we would expect that asymptotically exact  $O(1)$  behaviour corresponds to  $\tilde{S}(\vec{k}) \sim c_1 |\vec{k}|^2$  in the infinite system. This can be seen from (4.17) with  $\alpha^2$  identified with  $N$ . But  $\tilde{S}(\vec{k})$  for the complex zeros has the small  $|\vec{k}|$  behaviour (4.18) which is quartic in  $|\vec{k}|$ . Identifying  $\alpha^2$  with  $N$  in (4.19) then suggests that

$$\text{Var}(A) \sim \frac{c'}{N}. \quad (4.21)$$

We have not been able to prove this assertion in general, but it can be demonstrated in a specific example, using a combination of analytic and numerical analysis.

According to the formula (4.14), our task is to compute

$$\text{Var}(A) = \int_{\Omega} dS_1 \int_{\Omega} dS_2 a(\hat{r}_1) a(\hat{r}_2) \rho_{(2)}^T(\hat{r}_1, \hat{r}_2) + \int_{\Omega} dS a^2(\hat{r}) \rho_{(1)}(\hat{r}), \quad (4.22)$$

where  $dS$  is the differential surface element for the unit sphere,  $\rho_{(1)}(\hat{r})$  is given by (3.12) and

$$\rho_{(2)}^T(\hat{r}_1, \hat{r}_2) = \rho_{(2)}^T(\hat{r}_1 \cdot \hat{r}_2) \quad (4.23)$$

is given by (3.20). To proceed further, we suppose that in terms of the usual spherical coordinates  $(\theta, \phi)$ ,

$$a(\hat{r}) = a(\cos \theta) = a(\hat{r} \cdot \hat{z})$$

and thus depends only on the azimuthal angle. We want to use this functional dependence, together with the functional dependence displayed by (3.21) to simplify the double integral over the unit sphere in (4.22).

First consider

$$I_1 := \int_{\Omega} dS_1 a(\hat{r}_1 \cdot \hat{z}) \rho_{(2)}^T(\hat{r}_1 \cdot \hat{r}_2). \quad (4.24)$$

In this integral we change variables  $(\theta_1, \phi_1) \mapsto (\theta, \phi)$  by rotating the sphere so that  $\hat{r}_2 \mapsto \hat{z}$ , where now  $\theta$  is measured as if the direction  $\hat{r}_2$  were the  $z$ -axis and  $\phi = \phi_2$ . Then, in the new coordinate frame

$$\begin{aligned}\hat{r}_2 &= (0, 0, 1), \\ \hat{r}_1 &= (\cos \phi \sin \theta, \sin \phi \sin \theta, \cos \theta), \\ \hat{z} &= (-\cos \phi_2 \sin \theta_2, -\sin \phi_2 \sin \theta_2, \cos \theta_2).\end{aligned}$$

Thus

$$\begin{aligned}\hat{r} \cdot \hat{z} &= \cos \theta \cos \theta_2 - \cos \phi \cos \phi_2 \sin \theta \sin \theta_2 - \sin \phi \sin \phi_2 \sin \theta \sin \theta_2 \\ \hat{r}_1 \cdot \hat{r}_2 &= \cos \theta\end{aligned}$$

and so

$$I_1 = \int_{\Omega} dS a(\cos \theta \cos \theta_2 - \cos \phi \cos \phi_2 \sin \theta \sin \theta_2 - \sin \phi \sin \phi_2 \sin \theta \sin \theta_2) \rho_{(2)}^T(\cos \theta). \quad (4.25)$$

If we now further specialize and choose

$$a(\hat{r}_1 \cdot \hat{z}) = \hat{r}_1 \cdot \hat{z} \quad (4.26)$$

then, after recalling that

$$dS = \sin \theta d\theta d\phi, \quad (4.27)$$

we see that the integral over  $\phi$  in (4.25) can be carried out, leaving us with the single integral

$$I_1 = 2\pi \cos \theta_2 \int_0^\pi d\theta \sin \theta \cos \theta \rho_{(2)}^T(\cos \theta).$$

Substituting this simplification in (4.22), together with (4.26), (4.27) and (3.21) we thus obtain

$$\begin{aligned}\text{Var}\left(\sum_{j=1}^N \hat{r}_j \cdot \hat{z}\right) &= 4\pi^2 \int_0^\pi d\theta_2 \sin \theta_2 \cos^2 \theta_2 \int_0^\pi d\theta \sin \theta \cos \theta \rho_{(2)}^T(\cos \theta) \\ &\quad + \frac{N}{2} \int_0^\pi d\theta \sin \theta \cos^2 \theta \\ &= \frac{8\pi^2}{3} \int_0^\pi d\theta \sin \theta \cos \theta \rho_{(2)}^T(\cos \theta) + \frac{N}{3}.\end{aligned} \quad (4.28)$$

Although we have not been able to evaluate this integral analytically, the exact formula (3.20) makes numerical integration straightforward. The results are given in Table 1, and are consistent with the expected behaviour (4.21).

$N$	$\text{Var}(A)$	$N$	$\text{Var}(A)$
10	0.1249	80	0.0193
20	0.0704	90	0.0172
30	0.0489	100	0.0156
40	0.0375	110	0.0142
50	0.0303	120	0.0130
60	0.0255	130	0.0120
70	0.0220	140	0.0112

**Table 1.** Numerical values of (4.28), truncated at the fourth decimal, exhibiting the expected  $O(1/N)$  decay of  $\text{Var}(A)$ .

## 5 Correlations near the boundary of support

As remarked in the Introduction, to leading order the density of the complex zeros in the case of (1.6) is uniform in a disk of radius  $\sqrt{N}$  and zero outside that radius. The density of eigenvalues implied by the p.d.f. (1.7) have the same feature, and therefore so does the OCP at  $\beta = 2$  with the Boltzmann factor (1.7) taken to be valid throughout the plane (the harmonic potential between background and log-potential charge is only valid for the charge inside the background; outside the potential must be  $C + N \log |\vec{r}|$  by Newton's theorem). As the two-point function of the OCP has a universal form at the boundary, it is of some interest to compute the form of the one and two point functions for the complex zeros, and compare their behaviour to that of the OCP.

### 5.1 The density

In the case that the variance of the coefficients of the random polynomial (1.1) is given by (1.6), we have noted that the density of zeros is given by (3.7). Here we want to analyze  $\rho_{(1)}(\vec{r})$  with  $\vec{r}$  measured from the boundary of support of the density  $|\vec{r}| = \sqrt{N}$ . This can be achieved by writing

$$z = x + i(y - \sqrt{N}), \quad (5.29)$$

which sets the origin as the southern boundary of the support. According to (3.7) the asymptotic expansion of  $e(|z|^2; N)$  with  $z$  a function of  $N$  as specified by (5.29) is required. For this purpose we note from the definition (3.8) the easily verified formula

$$e(z; N) = e^z \frac{\Gamma(1 + N; z)}{\Gamma(1 + N)}, \quad \Gamma(N; a) := \int_a^\infty e^{-t} t^{N-1} dt,$$

and the asymptotic expansion [25]

$$\frac{\Gamma(1 + N; N + \sqrt{2N}u)}{\Gamma(1 + N)} = \frac{1}{2} \left( 1 - \text{erf}(u) + O\left(\frac{1}{\sqrt{N}}\right) \right). \quad (5.30)$$

Choosing  $u = (z - N)/\sqrt{2N}$  we see that

$$e(z; N) = \frac{1}{2} e^z \left( 1 - \text{erf}\left(\frac{z - N}{\sqrt{2N}}\right) + O\left(\frac{1}{\sqrt{N}}\right) \right), \quad (5.31)$$

valid for  $(z - N)/\sqrt{2N}$  bounded, which in turn implies

$$e(|x + i(y - \sqrt{N})|^2; N) \sim \frac{1}{2} e^{|z|^2} \left( 1 + \text{erf}(\sqrt{2}y) \right), \quad (5.32)$$

where  $z$  is given by (5.29). Hence, with the limiting form of  $\rho_{(1)}(\vec{r})$  denoted by  $\rho_{(1)}(y)$ , we see by substituting (5.32) in (3.7) that

$$\pi \rho_{(1)}(y) = 1 + \frac{1}{4} \frac{\partial^2}{\partial y^2} \log \left( 1 + \text{erf}(\sqrt{2}y) \right). \quad (5.33)$$

A plot of  $\pi\rho_{(1)}(y)$ , which is most expediently carried out by first explicitly calculating the second derivative to give

$$\pi\rho_{(1)}(y) = 1 - \frac{\sqrt{\frac{2}{\pi}}2ye^{-2y^2}(1 + \operatorname{erf}(\sqrt{2}y)) + \frac{2}{\pi}e^{-4y^2}}{(1 + \operatorname{erf}(\sqrt{2}y))^2}, \quad (5.34)$$

indicates rapid approaching of the bulk limit  $\pi\rho_{(1)}(y) = 1$  as  $y \rightarrow \infty$ , but slow decay outside the boundary of support as  $y \rightarrow -\infty$ . This observation is easily confirmed analytically from (5.33). Use of the expansion

$$1 + \operatorname{erf}(\sqrt{2}y) \sim \begin{cases} 1 + \frac{e^{-2y^2}}{\sqrt{2\pi}y}, & y \rightarrow \infty \\ \frac{e^{-2y^2}}{\sqrt{2\pi}|y|}, & y \rightarrow -\infty \end{cases}$$

shows

$$\pi\rho_{(1)}(y) \sim \begin{cases} 1 + 4\frac{1}{\sqrt{2\pi}}ye^{-2y^2}, & y \rightarrow \infty \\ \frac{1}{4y^2}, & y \rightarrow -\infty \end{cases} \quad (5.35)$$

Consider now the density for the OCP with Boltzmann factor proportional to (1.7). It is given by the exact formula [22]

$$\pi\rho_{(1)}(\vec{r}) = e^{-|\vec{r}|^2}e(|z|^2; N), \quad (5.36)$$

so we see from (5.32) that in the limit  $N \rightarrow \infty$  the density as measured from the southern most point on the boundary of support is given by

$$\pi\rho_{(1)}(y) = h(y), \quad h(y) := \frac{1}{2}(1 + \operatorname{erf}(\sqrt{2}y)). \quad (5.37)$$

In contrast to the edge density (5.34) which exhibits algebraic decay as seen in (5.35) for large distances outside the boundary of support, here the asymptotic values are reached with only Gaussian corrections in both directions,

$$\pi\rho_{(1)}(y) \sim \begin{cases} 1 - \frac{e^{-2y^2}}{2\sqrt{2\pi}|y|}, & y \rightarrow \infty \\ \frac{e^{-2y^2}}{2\sqrt{2\pi}|y|}, & y \rightarrow -\infty \end{cases}$$

The sharpness of the boundary of support for the density associated with (1.7), as generated from the eigenvalues of complex random matrices, relative to that for the zeros of complex random matrices can be illustrated by plotting the zeros for a particular realization in each case. This is done in Figure 1.

Another point of interest is that analogous behaviour to (5.35) is found for the OCP at  $\beta = 2$  with the correct Coulomb potential  $C + N \log r$  between particles and background outside the disk. The exact density as measured from the southern most boundary of the uniform background is given by [27]

$$\rho_{(1)}(y) = f(y) \frac{2}{\sqrt{\pi}} \int_0^\infty \frac{\exp(2^{3/2}tx - t^2)}{1 + \operatorname{erf}(t) + e^{-t^2}/t\sqrt{\pi}} \quad (5.38)$$

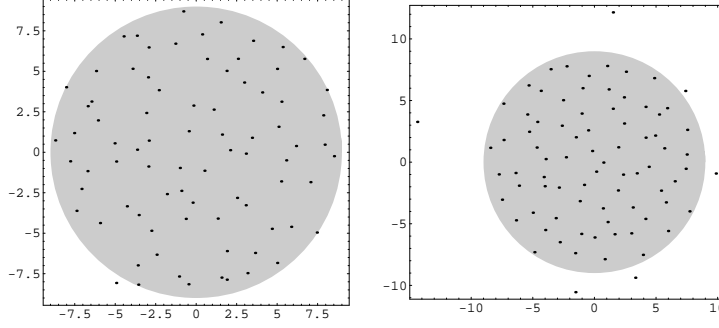


Figure 1: A typical realization of the eigenvalues of a  $81 \times 81$  complex Gaussian random matrix (leftmost plot) and the zeros of a complex Gaussian random polynomial of degree 81 with variances given by (1.6). The shaded region represents the disk  $|z| < 9$  which is the leading order support of the density in both cases. Outside the disk the density has a  $1/r^2$  tail in the case of the zeros, whereas it falls off as a Gaussian for the eigenvalues, in keeping with the realizations in the Figure.

where  $f(y) = e^{-2y^2}$ ,  $y > 0$  and  $f(y) = 1$ ,  $y < 0$ . This gives for the asymptotic behaviour [27]

$$\pi\rho_{(1)}(y) \sim \begin{cases} 1 - \frac{e^{-2y^2}}{2\sqrt{2\pi}|y|^3}, & y \rightarrow \infty \\ \frac{1}{4y^2}, & y \rightarrow -\infty \end{cases} \quad (5.39)$$

An analytic feature of the surface density for the OCP in general is that the total charge is zero. This means

$$\int_{-\infty}^0 \pi\rho_{\text{OCP}(1)}(y) dy + \int_0^{\infty} \pi(\rho_{\text{OCP}(1)}(y) - 1) dy = 0, \quad (5.40)$$

which indeed can be verified for the OCP profiles (5.36) and (5.38). A straightforward calculation using (5.33) also shows that (5.40) holds true for the boundary density of the complex zeros.

## 5.2 The two-point distribution

Let  $\rho_{(2)}(y_1, y_2; x_1 - x_2)$  denote the  $N \rightarrow \infty$  limit of  $\rho_{(2)}(\vec{r}_1, \vec{r}_2)$  as specified by (3.16) with

$$z_1 = x_1 + i(y_1 - \sqrt{N}), \quad z_2 = x_2 + i(y_2 - \sqrt{N}). \quad (5.41)$$

The basic strategy is to explicitly compute the limiting value of the matrix elements of

$$U := dC - B^\dagger A^D B \quad (5.42)$$

and the determinant  $d$  in (3.16). From (3.15) and (3.17) we have

$$d = a_{11}a_{22} - a_{12}a_{21} = e(|z_1|^2; N)e(|z_2|^2; N) - |e(z_1\bar{z}_2; N)|^2.$$

With  $z_1$  and  $z_2$  given by (5.41) we see that the asymptotics of the first term follows from (5.32), while (5.31) shows

$$e(z_1 \bar{z}_2; N) \sim \frac{1}{2} e^{z_1 \bar{z}_2} \left\{ 1 + \operatorname{erf} \left( \frac{1}{\sqrt{2}} (y_1 + y_2 - i(x_1 - x_2)) \right) \right\}. \quad (5.43)$$

Thus, with  $h(y)$  defined as in (5.37) we have

$$\begin{aligned} d \sim & e^{2N-2\sqrt{N}(y_1+y_2)} e^{(x_1^2+x_2^2+y_1^2+y_2^2)} \left( h(y_1) h(y_2) \right. \\ & \left. - e^{-(x_1-x_2)^2-(y_1-y_2)^2} \left| h \left( \frac{1}{2} (y_1 + y_2 - i(x_1 - x_2)) \right) \right|^2 \right). \end{aligned} \quad (5.44)$$

The analysis of the leading order behaviour of the matrix elements of (5.42) is more involved. Each element consists of the sum of many individual elements, and we find that a naive leading order expansion of each term individually gives zero, as all such contributions exactly cancel. It is therefore necessary to expand individual terms to higher order. This can conveniently be achieved by writing the terms  $e(z; N + p)$  which occur therein with  $p \neq 0$  in terms of  $e(z; N)$ . For example, from the definition (3.8) we have

$$e(z; N - 1) = e(z; N) - \frac{z^N}{N!}, \quad e(z; N - 2) = e(z; N) - \frac{z^N}{N!} - \frac{z^{N-1}}{(N-1)!} \quad (5.45)$$

which we use to substitute for  $e(z; N - 1)$  and  $e(z; N - 2)$ .

Let us give the details of the implementation of this procedure in the case of  $u_{11}$ . First, according to (5.42), we have

$$u_{11} = dc_{11} + \bar{b}_{11}a_{12}b_{21} + \bar{b}_{21}a_{21}b_{11} - \bar{b}_{11}a_{22}b_{11} - \bar{b}_{21}a_{11}b_{21}.$$

Substituting in the explicit form of the matrix elements according to (3.17) then gives

$$\begin{aligned} u_{11} = & \left( |z_1|^2 e(|z_1|^2; N - 2) + e(|z_1|^2; N - 1) \right) \left( e(|z_1|^2; N) e(|z_2|^2; N) - |e(z_1 \bar{z}_2; N)|^2 \right) \\ & - |z_1|^2 |e(|z_1|^2; N - 1)|^2 e(|z_2|^2; N) - |z_2|^2 |e(z_1 \bar{z}_2; N - 1)|^2 e(|z_1|^2; N) \\ & + 2\operatorname{Re} \left( z_1 \bar{z}_2 e(z_1 \bar{z}_2; N - 1) e(\bar{z}_1 z_2; N) e(|z_1|^2; N - 1) \right). \end{aligned}$$

With the substitution (5.45) this can be rewritten to read

$$u_{11} = (a) + (b) + (c) + (d) + (e) + (f) + (g1) + (g2) + (g3),$$

where

$$\begin{aligned} (a) &:= |e(|z_1|^2; N)|^2 e(|z_2|^2; N) \\ (b) &:= |e(z_1 \bar{z}_2; N)|^2 e(|z_1|^2; N) \left( -1 - |z_1|^2 - |z_2|^2 + z_1 \bar{z}_2 + z_2 \bar{z}_1 \right) \\ (c) &:= e(|z_1|^2; N) e(|z_2|^2; N) \left( -\frac{|z_1|^{2N}}{N!} + \frac{|z_1|^{2N}}{N!} (|z_1|^2 - N) \right) \\ (d) &:= |e(z_1 \bar{z}_2; N)|^2 \frac{|z_1|^{2N}}{N!} \left( |z_1|^2 + N + 1 - z_1 \bar{z}_2 - \bar{z}_1 z_2 \right) \end{aligned}$$

$$\begin{aligned}
(e) &:= e(|z_1|^2; N) e(z_1 \bar{z}_2; N) \frac{(\bar{z}_1 z_2)^N}{N!} (|z_2|^2 - \bar{z}_1 z_2) \\
(f) &:= e(|z_1|^2; N) e(\bar{z}_1 z_2; N) \frac{(z_1 \bar{z}_2)^N}{N!} (|z_2|^2 - z_1 \bar{z}_2) \\
(g1) &:= -\frac{|z_1|^{4N+2}}{(N!)^2} e(|z_2|^2; N) \\
(g2) &:= -\frac{|z_1|^{2N} |z_2|^{2N+2}}{(N!)^2} e(|z_1|^2; N) \\
(g3) &:= 2 \frac{|z_1|^{2N}}{N!} \operatorname{Re} \left( \frac{(z_1 \bar{z}_2)^{N+1}}{N!} e(\bar{z}_1 z_2; N) \right).
\end{aligned}$$

It now remains to calculate the large  $N$  asymptotic behaviour of each of these terms with  $z_1$  and  $z_2$  given by (5.41). Using (5.32), (5.43) and the expansion

$$\begin{aligned}
\frac{(z_i \bar{z}_j)^N}{N!} &\sim \frac{1}{\sqrt{2\pi N}} \exp \left( N - \sqrt{N}(y_i + y_j + i(x_j - x_i)) \right. \\
&\quad \left. + \frac{1}{2}(x_i^2 + x_j^2 - y_i^2 - y_j^2) + i(x_i y_i - x_j y_j) \right)
\end{aligned}$$

which follows from the expansion  $(1+u)^N \sim e^{N(u-u^2/2+\dots)}$  and Stirling's formula, we obtain

$$\begin{aligned}
(a) &\sim e^{3N-4\sqrt{N}y_1-2\sqrt{N}y_2} \exp \left( 2(x_1^2 + y_1^2) + x_2^2 + y_2^2 \right) h^2(y_1) h(y_2) \\
(b) &\sim e^{3N-4\sqrt{N}y_1-2\sqrt{N}y_2} \left( -(x_1 - x_2)^2 - (y_1 - y_2)^2 - 1 \right) \exp \left( x_1^2 + y_1^2 + 2x_1 x_2 + 2y_1 y_2 \right) \\
&\quad \times h(y_1) \left| h \left( \frac{1}{2}(y_1 + y_2 + i(x_1 - x_2)) \right) \right|^2 \\
(c) &\sim -e^{3N-4\sqrt{N}y_1-2\sqrt{N}y_2} \sqrt{\frac{2}{\pi}} y_1 \exp(2x_1^2 + x_2^2 + y_2^2) h(y_1) h(y_2) \\
(d) &\sim e^{3N-4\sqrt{N}y_1-2\sqrt{N}y_2} \sqrt{\frac{2}{\pi}} y_2 \exp(x_1^2 - y_1^2 + 2(x_1 x_2 + y_1 y_2)) \\
&\quad \times \left| h \left( \frac{1}{2}(y_1 + y_2 + i(x_1 - x_2)) \right) \right|^2 \\
(e) + (f) &\sim e^{3N-4\sqrt{N}y_1-2\sqrt{N}y_2} \exp \left( \frac{1}{2}(3x_1^2 + y_1^2 + x_2^2 - y_2^2) + x_1 x_2 + y_1 y_2 \right) h(y_1) \\
&\quad \times 2 \operatorname{Re} \left\{ \frac{1}{\sqrt{2\pi}} (y_1 - y_2 + i(x_1 - x_2)) \exp \left( i(x_2 y_2 - x_1 y_1 + x_2 y_1 - x_1 y_2) \right) \right. \\
&\quad \left. \times h \left( \frac{1}{2}(y_1 + y_2 - i(x_1 - x_2)) \right) \right\} \\
(g1) &\sim -e^{3N-4\sqrt{N}y_1-2\sqrt{N}y_2} \frac{1}{2\pi} \exp(2x_1^2 - 2y_1^2 + x_2^2 + y_2^2) h(y_2) \\
(g2) &\sim -e^{3N-4\sqrt{N}y_1-2\sqrt{N}y_2} \frac{1}{2\pi} \exp(2x_1^2 + x_2^2 - y_2^2) h(y_1) \\
(g3) &\sim e^{3N-4\sqrt{N}y_1-2\sqrt{N}y_2} \frac{1}{2\pi} \exp \left( \frac{1}{2}(3x_1^2 - 3y_1^2 + x_2^2 - y_2^2) + x_1 x_2 + y_1 y_2 \right) \\
&\quad \times 2 \operatorname{Re} \left\{ \exp \left( i(x_1 y_1 - x_2 y_2 + x_1 y_2 - x_2 y_1) \right) h \left( \frac{1}{2}(y_1 + y_2 + i(x_1 - x_2)) \right) \right\}.
\end{aligned}$$

Notice that all of the above asymptotic forms contain an  $N$  dependent factor  $e^{3N-4\sqrt{N}y_1-2\sqrt{N}y_2}$ . Similarly decomposing  $u_{22}$  shows that the asymptotic form of each term contains the  $N$  dependent factor  $e^{3N-4\sqrt{N}y_2-2\sqrt{N}y_1}$ . Now, according to (3.15) and (3.16)

$$\pi^2 \rho_{(2)}(\vec{r}_1, \vec{r}_2) = \frac{u_{11} u_{22}}{d^3} + \frac{u_{12} u_{21}}{d^3}$$

so we are required to form the product  $u_{11}u_{22}$  and to divide by  $d^3$ . Doing this, we see that the  $N$  dependent factors cancel out as expected. Similarly we find that  $N$  dependent factors in  $u_{12}u_{21}/d^3$  cancel. Collecting together the remaining terms, which are calculated for  $u_{22}$ ,  $u_{12}$  and  $u_{21}$  in the same manner as detailed above for  $u_{11}$ , we obtain for the limiting value of the edge two-point distribution

$$\pi^2 \rho_{(2)}(y_1, y_2; x_1 - x_2) = \frac{v_{11}v_{22}}{d_1^3} + \frac{v_{12}v_{21}}{d_2^3} \quad (5.46)$$

where

$$d_1 := h(y_1)h(y_2) - e^{-(x_1-x_2)^2-(y_1-y_2)^2} \left| h\left(\frac{1}{2}(y_1 + y_2 + i(x_1 - x_2))\right) \right|^2 \quad (5.47)$$

$$d_2 := e^{(x_1-x_2)^2+(y_1-y_2)^2} d_1 \quad (5.48)$$

$$\begin{aligned} v_{11} = & h^2(y_1)h(y_2) - \left( (x_1 - x_2)^2 + (y_1 - y_2)^2 + 1 \right) e^{-(x_1-x_2)^2-(y_1-y_2)^2} h(y_1) \\ & \times \left| h\left(\frac{1}{2}(y_1 + y_2 + i(x_1 - x_2))\right) \right|^2 - \sqrt{\frac{2}{\pi}} y_1 e^{-2y_1^2} h(y_1)h(y_2) \\ & + \sqrt{\frac{2}{\pi}} y_2 e^{-2y_2^2} e^{-(x_1-x_2)^2-(y_1-y_2)^2} \left| h\left(\frac{1}{2}(y_1 + y_2 + i(x_1 - x_2))\right) \right|^2 \\ & + 2\text{Re}\left\{ \frac{1}{\sqrt{2\pi}} (y_1 - y_2 + i(x_1 - x_2)) e^{-(y_1^2+y_2^2)} e^{-(x_1-x_2)^2/2-(y_1-y_2)^2/2} \right. \\ & \times e^{i(x_2-x_1)(y_1+y_2)} h(y_1)h\left(\frac{1}{2}(y_1 + y_2 - i(x_1 - x_2))\right) \left. \right\} \\ & - \frac{1}{2\pi} e^{-4y_1^2} h(y_2) - \frac{1}{2\pi} e^{-2(y_1^2+y_2^2)} h(y_1) + 2\text{Re}\left\{ \frac{1}{2\pi} \right. \\ & \times e^{-3y_1^2-y_2^2} e^{-(x_2-x_1)^2/2-(y_2-y_1)^2/2} e^{i(x_1-x_2)(y_1+y_2)} h\left(\frac{1}{2}(y_1 + y_2 + i(x_1 - x_2))\right) \left. \right\} \end{aligned} \quad (5.49)$$

$$v_{22} = v_{11} \Big|_{\substack{x_1 \leftrightarrow x_2 \\ y_1 \leftrightarrow y_2}} \quad (5.50)$$

$$\begin{aligned} v_{12} = & -\left( (x_1 - x_2)^2 + (y_1 - y_2)^2 - 1 \right) e^{(x_1-x_2)^2+(y_1-y_2)^2} h(y_1)h(y_2)h\left(\frac{1}{2}(y_1 + y_2 - i(x_1 - x_2))\right) \\ & - h\left(\frac{1}{2}(y_1 + y_2 - i(x_1 - x_2))\right) \left| h\left(\frac{1}{2}(y_1 + y_2 + i(x_1 - x_2))\right) \right|^2 \\ & - \frac{1}{\sqrt{2\pi}} (y_1 + y_2 + i(x_1 - x_2)) e^{-(y_1^2+y_2^2)} e^{3(x_2-x_1)^2/2+3(y_2-y_1)^2/2} e^{i(x_1-x_2)(y_1+y_2)} h(y_1)h(y_2) \\ & + \frac{1}{\sqrt{2\pi}} (y_1 + y_2 - i(x_1 - x_2)) e^{-(y_1^2+y_2^2)} e^{(x_1-x_2)^2/2+(y_1-y_2)^2/2} \\ & \times e^{i(x_1-x_2)(y_1+y_2)} \left| h\left(\frac{1}{2}(y_1 + y_2 + i(x_1 - x_2))\right) \right|^2 \\ & + \frac{1}{\sqrt{2\pi}} (y_1 - y_2 + i(x_1 - x_2)) e^{-2y_2^2} e^{(x_2-x_1)^2+(y_2-y_1)^2} h(y_1)h\left(\frac{1}{2}(y_1 + y_2 - i(x_1 - x_2))\right) \\ & + \frac{1}{\sqrt{2\pi}} (y_2 - y_1 + i(x_1 - x_2)) e^{-2y_2^2} e^{(x_2-x_1)^2+(y_2-y_1)^2} h(y_2)h\left(\frac{1}{2}(y_1 + y_2 - i(x_1 - x_2))\right) \\ & + \frac{1}{2\pi} e^{-2(y_1^2+y_2^2)} e^{(x_2-x_1)^2+(y_2-y_1)^2} e^{2i(x_1-x_2)(y_1+y_2)} h\left(\frac{1}{2}(y_1 + y_2 + i(x_1 - x_2))\right) \\ & + \frac{1}{2\pi} e^{-2(y_1^2+y_2^2)} e^{(x_2-x_1)^2+(y_2-y_1)^2} h\left(\frac{1}{2}(y_1 + y_2 - i(x_1 - x_2))\right) \\ & - \frac{1}{2\pi} e^{-3y_1^2-y_2^2} e^{3(x_2-x_1)^2/2+3(y_2-y_1)^2/2} e^{i(x_1-x_2)(y_1+y_1)} h(y_2) \\ & - \frac{1}{2\pi} e^{-y_1^2-3y_2^2} e^{3(x_2-x_1)^2/2+3(y_2-y_1)^2/2} e^{i(x_1-x_2)(y_1+y_1)} h(y_1) \end{aligned} \quad (5.51)$$

$$v_{21} = \bar{v}_{12}. \quad (5.52)$$

The above explicit forms show that indeed  $\pi^2 \rho_{(2)}(y_1, y_2; x_1 - x_2)$  as specified by (5.46) is a function of  $x_1 - x_2$  as claimed in the notation. Physically, this corresponds to translation invariance in the direction of the boundary. Another check on (5.46) is that in the limit  $y_1, y_2 \rightarrow \infty$ ,  $y_1 - y_2$  fixed, the bulk result (3.18) is reclaimed. In this limit, since  $\text{erf}(u) \rightarrow 1$  as  $u \rightarrow \infty$  we have  $h(u) \rightarrow 1$  as  $u \rightarrow \infty$ . This shows that

$$\begin{aligned} v_{11} \sim v_{22} &\sim 1 - (r^2 + 1)e^{-r^2}, & v_{12} \sim v_{21} &\sim -(r^2 - 1)e^{r^2} - 1 \\ d_1 &\sim 1 - e^{-r^2}, & d_2 &\sim e^{r^2} - 1 \end{aligned} \quad (5.53)$$

where  $r^2 := (x_1 - x_2)^2 + (y_1 - y_2)^2$ , and hence

$$\pi^2 \rho_{(2)}(y_1, y_2; x_1 - x_2) \sim \frac{1}{(1 - e^{-r^2})^3} \left( e^{-2r^2} (e^{r^2} - 1 - r^2)^2 + e^{-r^2} (e^{-r^2} - 1 + r^2)^2 \right)$$

which is (3.18) in the form given by Prosen [10, eq. (27)].

It is of interest to compare the structure of (5.46) with the limiting value of the edge two-point distribution function for the OCP with Boltzmann factor (1.7). In the finite system the exact two-point distribution is given by [22]

$$\pi^2 \rho_{(2)}(\vec{r}_1, \vec{r}_2) = \pi^2 \rho_{(1)}(\vec{r}_1) \rho_{(1)}(\vec{r}_2) - \pi^2 e^{-|\vec{r}_1|^2 - |\vec{r}_2|^2} |e(z_1 \bar{z}_2; N)|^2 \quad (5.54)$$

where  $\pi \rho_{(1)}(\vec{r})$  is specified by (5.36). Use of (5.43) then gives that in the limit  $N \rightarrow \infty$  with the coordinates (5.41) we have

$$\pi^2 \rho_{(2)}(y_1, y_2; x_1 - x_2) = \pi^2 \rho_{(1)}(y_1) \rho_{(1)}(y_2) - \pi^2 e^{-(x_1 - x_2)^2 - (y_1 - y_2)^2} \left| \rho_{(1)} \left( \frac{1}{2} (y_1 + y_2 - i(x_1 - x_2)) \right) \right|^2 \quad (5.55)$$

where  $\pi \rho_{(1)}(y)$  is specified by (5.37). In (5.55) the general structure of the two-point distribution, as a product of the one-body densities plus a correction term (the truncated two-point distribution, or two-point correlation function  $\rho_{(2)}^T$ ) which decays as the points  $(x_1, y_1)$  and  $(x_2, y_2)$  are separated, is displayed. Thus we have

$$\rho_{(2)}^T(y_1, y_2; x_1 - x_2) = -e^{-(x_1 - x_2)^2 - (y_1 - y_2)^2} \left| \rho_{(1)} \left( \frac{1}{2} (y_1 + y_2 - i(x_1 - x_2)) \right) \right|^2. \quad (5.56)$$

This is distinct from the exact expression (5.46) in which such a decomposition is not exhibited.

### 5.3 Asymptotics of the two-point correlation

In this subsection the asymptotic behaviour of the two-point correlation parallel to the boundary of support of the density will be investigated. In the case of the two-point correlation (5.55) for the OCP at  $\beta = 2$ , this behaviour must conform to a universal form obeyed by the asymptotics of the charge-charge correlation parallel to a plane boundary for general two-dimensional Coulomb systems in their conducting phase. This asymptotic behaviour in the case of the OCP specifies that [28]

$$\rho_{\text{OCP}(2)}^T(y_1, y_2; x_1 - x_2) \underset{|x_1 - x_2| \rightarrow \infty}{\sim} -\frac{f(y_1, y_2)}{2\beta\pi^2(x_1 - x_2)^2}, \quad \int_{-\infty}^{\infty} dy_1 \int_{-\infty}^{\infty} dy_2 f(y_1, y_2) = 1. \quad (5.57)$$

To test the prediction (5.57) on the exact result (5.55), the asymptotic expansion of the error function  $\text{erf}(Y + iX)$  for large imaginary argument is required. Now, by choosing the path of integration from 0 to  $Y + iX$  in the definition of  $\text{erf}(Y + iX)$  to first go along the imaginary axis to  $iX$  then parallel to the real axis to  $Y + iX$ , we see that it is possible to write

$$\text{erf}(Y + iX) = i \frac{2}{\sqrt{\pi}} \int_0^X e^{s^2} ds + \frac{2}{\sqrt{\pi}} e^{X^2} \int_0^Y e^{-2iXs - s^2} ds.$$

Using integration by parts we can then deduce that

$$\text{erf}(Y + iX) \underset{X \rightarrow \infty}{\sim} \frac{1}{\sqrt{\pi}} e^{X^2 - Y^2 - 2iXY} \left( \frac{i}{X} - \frac{Y}{X^2} + \frac{i}{2X^3} (1 - 2Y^2) - \frac{Y(3 - 2Y^2)}{2X^4} + O\left(\frac{1}{X^5}\right) \right). \quad (5.58)$$

This in turn, when used in (5.55), implies

$$\rho_{\text{OCP}(2)}^T(y_1, y_2; x_1 - x_2) \underset{|x_1 - x_2| \rightarrow \infty}{\sim} -\frac{f(y_1)f(y_2)}{4\pi^2(x_1 - x_2)^2}, \quad f(y) := \frac{d}{dy} \pi \rho_{(1)}(y) \quad (5.59)$$

which indeed obeys (5.57).

Let us now consider the asymptotic expansion of (5.46) for  $|x_1 - x_2| \rightarrow \infty$ . Since this expression represents the full two-point distribution rather than its truncated counterpart we expect the leading behaviour to be given by  $\pi^2 \rho_{(1)}(y_1) \rho_{(1)}(y_2)$ , where each  $\rho_{(1)}(y)$  is specified by (5.34). The next term will then give the leading order behaviour of  $\rho_{(2)}^T$ . Now according to (5.46) and (5.48), (5.50), (5.52) the asymptotic behaviour follows from the behaviour of the quantities  $d_1$ ,  $v_{11}$  and  $v_{12}$ . Inspection of (5.47), (5.49) and (5.51) shows that the latter asymptotic behaviour follows from that of  $h(\frac{1}{2}(y_1 + y_2 - i(x_1 - x_2)))$ , its complex conjugate and its modulus squared. Now, from the definition (5.37) and the expansion (5.58) we have that

$$\begin{aligned} h\left(\frac{1}{2}(y_1 + y_2 - i(x_1 - x_2))\right) &\sim \frac{1}{\sqrt{2\pi}} e^{\frac{1}{2}(x_1 - x_2)^2 - \frac{1}{2}(y_1 + y_2)^2 - i(x_2 - x_1)(y_1 + y_2)} \\ &\quad \times \left( \frac{i}{x_2 - x_1} - \frac{y_1 + y_2}{(x_2 - x_1)^2} + \frac{i}{(x_2 - x_1)^3} (1 - (y_1 + y_2)^2) + \dots \right) \\ h\left(\frac{1}{2}(y_1 + y_2 + i(x_1 - x_2))\right) &\sim h\left(\frac{1}{2}(y_1 + y_2 - i(x_1 - x_2))\right) \\ \left| h\left(\frac{1}{2}(y_1 + y_2 - i(x_1 - x_2))\right) \right|^2 &\sim \frac{1}{2\pi} e^{(x_1 - x_2)^2 - (y_1 + y_2)^2} \left( \frac{1}{(x_1 - x_2)^2} \right. \\ &\quad \left. + \frac{1}{(x_1 - x_2)^4} (2 - (y_1 + y_2)^2) + \dots \right) \end{aligned}$$

After some calculation we then find

$$d_1 \sim h(y_1)h(y_2) \left( 1 - \frac{e^{-2(y_1^2 + y_2^2)}}{2\pi h(y_1)h(y_2)} \frac{1}{(x_1 - x_2)^2} \right) \quad (5.60)$$

$$\begin{aligned} v_{11} &\sim h(y_2) \left( h^2(y_1) - \sqrt{\frac{2}{\pi}} y_1 e^{-2y_1^2} h(y_1) - \frac{1}{2\pi} e^{-4y_1^2} \right) \\ &\quad + \frac{e^{-2(y_1^2 + y_2^2)}}{(x_1 - x_2)^2} \left( -\frac{1}{2\pi} h(y_1)(1 + 4y_1^2) - \frac{1}{\sqrt{2\pi^3}} y_1 e^{-2y_1^2} \right) \end{aligned} \quad (5.61)$$

$$\begin{aligned} v_{12} &\sim e^{\frac{3}{2}(x_1 - x_2)^2} e^{-\frac{3}{2}(y_1 + y_2)^2} e^{-i(x_2 - x_1)(y_1 + y_2)} \\ &\quad \left\{ \frac{i}{x_2 - x_1} \left( \frac{1}{\sqrt{2\pi}} 4y_1 y_2 h(y_1) h(y_2) e^{2(y_1^2 + y_2^2)} + \frac{1}{\pi} y_1 h(y_1) e^{2y_1^2} + \frac{1}{\pi} y_2 h(y_2) e^{2y_2^2} + \frac{1}{\sqrt{8\pi^3}} \right) \right. \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{(x_2 - x_1)^2} \left( \frac{2}{\sqrt{2\pi}} (y_1 + y_2) (1 - 2y_1 y_2) h(y_1) h(y_2) e^{2(y_1^2 + y_2^2)} + \frac{1}{2\pi} (1 - 2y_2(y_1 + y_2)) \right. \\
& \times h(y_2) e^{2y_2^2} + \frac{1}{2\pi} (1 - 2y_1(y_1 + y_2)) h(y_1) e^{2y_1^2} - \frac{1}{\sqrt{8\pi^3}} (y_1 + y_2) \Big) \Big\} \quad (5.62)
\end{aligned}$$

Substituting the results (5.60), (5.61) and (5.62) in (5.46), recalling the relations (5.48), (5.50) and (5.52), and performing some further calculation we find the sought expansion,

$$\pi^2 \rho_{(2)}(y_1, y_2; x_1 - x_2) \sim \frac{g(y_1)}{h^2(y_1)} \frac{g(y_2)}{h^2(y_2)} + \frac{Q(y_1, y_2)}{(x_1 - x_2)^2} \quad (5.63)$$

where

$$g(y) := h^2(y) - \sqrt{\frac{2}{\pi}} y e^{-2y^2} h(y) - \frac{1}{2\pi} e^{-4y^2}$$

$$\begin{aligned}
Q(y_1, y_2) &:= \frac{1}{h^3(y_1) h^3(y_2)} \left\{ \frac{1}{\pi} (8y_1^2 y_2^2 - 2y_1^2 - 2y_2^2 + \frac{1}{2}) h^2(y_1) h^2(y_2) e^{-2(y_1^2 + y_2^2)} \right. \\
&+ \frac{1}{\sqrt{2\pi^3}} \left( (12y_1^2 y_2 - 3y_2) h^2(y_1) h(y_2) e^{2y_1^2} + (12y_1 y_2^2 - 3y_1) h(y_1) h^2(y_2) e^{2y_2^2} \right) \\
&\times e^{-4(y_1^2 + y_2^2)} + \frac{9}{\pi^2} y_1 y_2 h(y_1) h(y_2) e^{-4(y_1^2 + y_2^2)} \\
&+ \frac{1}{2\pi^2} \left( (4y_1^2 - 1) h^2(y_1) e^{-2(y_1^2 + 3y_2^2)} + (4y_2^2 - 1) h^2(y_2) e^{-2(3y_1^2 + y_2^2)} \right) \\
&+ \frac{1}{\sqrt{8\pi^5}} \left( 6y_1 h(y_1) e^{-2(2y_1^2 + 3y_2^2)} + 6y_2 h(y_2) e^{-2(3y_1^2 + 2y_2^2)} \right) \\
&\left. + \frac{1}{2\pi^3} e^{-6(y_1^2 + y_2^2)} \right\} \quad (5.64)
\end{aligned}$$

Comparison with (5.34) shows that indeed the leading order behaviour is  $\pi^2 \rho_{(1)}(y_1) \rho_{(1)}(y_2)$  so we have

$$\rho_{(2)}^T(y_1, y_2; x_1 - x_2) \sim \frac{Q(y_1, y_2)}{4\pi^2 (x_1 - x_2)^2}.$$

Furthermore, a close inspection of (5.64) and (5.34) reveals that it factorizes as

$$Q(y_1, y_2) = f(y_1) f(y_2), \quad f(y) := \frac{d}{dy} \pi \rho_{(1)}(y).$$

Consequently we have

$$\rho_{(2)}^T(y_1, y_2; x_1 - x_2) \sim \frac{f(y_1) f(y_2)}{4\pi^2 (x_1 - x_2)^2}, \quad \int_{-\infty}^{\infty} f(y) dy = 1, \quad (5.65)$$

which is identical to the behaviour (5.57) for the OCP at  $\beta = 2$ , except for the sign.

Note also that since for small separation between  $(x_1, y_1)$  and  $(x_2, y_2)$  the truncated distribution is negative, the positive tail for large  $|x_1 - x_2|$  exhibited by (5.57) indicates that  $\rho_{(2)}^T$  changes sign, analogous to its behaviour in the bulk. In contrast the same quantity for the OCP at  $\beta = 2$  is always negative, as seen from (5.56).

## 6 Summary

The p.d.f. (1.3) when interpreted as a Boltzmann factor consists of the pairwise logarithmic interaction (1.5) as well as an extensive many body interaction. If the extensive many body interaction is replaced by a one body Gaussian factor, the p.d.f. becomes identical to the p.d.f. (1.7) giving the eigenvalue distribution for complex Gaussian random matrices, or equivalently the Boltzmann factor for a 2dOCP. The exact distribution functions for both (1.3) and (1.7) can be calculated exactly, enabling a comparative study of the two systems to be undertaken.

Some similarities, and some differences, have been found. The similarities include the same  $y \rightarrow -\infty$  decay of the density profile away from the boundary being displayed by (5.35) for the complex zeros, and by (5.39) for the soft wall OCP. Also both systems have the same (up to a minus sign) universal form (5.57) of the two-point correlation parallel to the boundary. A notable difference is that while the second moment of the two-point function in the bulk of the OCP obeys the Stillinger-Lovett sum rule (4.9), the same quantity for the complex zeros vanishes. This was shown to have consequence regarding the behaviour of the variance of a linear statistic, a quantity which has received recent attention in the context of two-dimensional Coulomb systems [17], and the zeros of multidimensional random polynomials with a constant density on the surface of higher dimensional spheres [14].

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